Final Exam — Ordinary Differential Equations (WIGDV–07)

Wednesday 28 October 2015, 14.00h–17.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (12 points)

Solve the following initial value problem:

$$y' = \frac{y}{x}\big(1 + \log(y) - \log(x)\big), \qquad y(1) = e.$$

Problem 2 (10 points)

Solve the following Bernoulli equation:

$$\frac{dy}{dx} + 2xy + xy^4 = 0.$$

Problem 3 (2 + 12 + 6 points)

Consider the following 3×3 matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

(a) Show that $\det(A - \lambda I) = (1 - \lambda)^3$

(b) Compute the Jordan canonical form of A.

(c) Compute a fundamental matrix for the system $\mathbf{y}' = A\mathbf{y}$.

Problem 4 (5 + 5 + 5 + 5 points)

Let $b > 0$ be arbitrary, and let $C([0, b])$ denote the space of continuous functions on the interval $[0, b]$. For all $\alpha > 0$ the norm

$$\|y\| = \sup\left\{|y(x)|e^{-\alpha x} \,:\, x \in [0, b]\right\}$$

turns $C([0, b])$ into a Banach space. Consider the integral operator

$$T : C([0, b]) \to C([0, b]), \qquad (Ty)(x) = \int_0^x \sqrt{1 + y(t)^2}\, dt.$$

Prove the following statements:

(a) $\left|\sqrt{1 + y^2} - \sqrt{1 + z^2}\right| \leq |y - z| \qquad \forall\, y, z \in \mathbb{R}.$

(b) $\left|(Ty)(x) - (Tz)(x)\right| \leq \dfrac{e^{\alpha x} - 1}{\alpha}\|y - z\| \qquad \forall\, y, z \in C([0, b]),\ x \in [0, b].$

(c) $\|Ty - Tz\| \leq \dfrac{1}{\alpha}\|y - z\| \qquad \forall\, y, z \in C([0, b])$

(d) The initial value problem

$$y' = \sqrt{1 + y^2}, \qquad y(0) = 0.$$

has a unique solution on the interval $[0, b]$.

Problem 5 (3 + 10 points)

Consider the second-order equation:

$$u'' - 4xu' + (4x^2 - 2)u = 0$$

(a) Show that $u_1(x) = e^{x^2}$ is a solution.

(b) Compute a second solution of the form $u_2(x) = c(x)e^{x^2}$ such that u_1 and u_2 are linearly independent.

Problem 6 (10 + 5 points)

Consider the semi-homogeneous boundary value problem

$$\frac{d^2u}{dx^2} - u = f(x), \qquad u(0) = 0, \qquad u(1) = 0.$$

where $f(x)$ is a continuous function.

(a) Compute Green's function.

(b) Solve the boundary value problem with $f(x) = e^{-x}$ using Green's function.

End of test (90 points)

Solution of Problem 1 (12 points)

We can rewrite the differential equation as follows:

$$\frac{dy}{dx} = \frac{y}{x} + \frac{y}{x}\log\left(\frac{y}{x}\right)$$

Taking the substitution $u = y/x$ gives the new differential equation

$$\frac{du}{dx} = \frac{u\log(u)}{x}$$

(3 points)

Separating the variables gives

$$\int \frac{1}{u\log(u)}\,du = \int \frac{1}{x}\,dx$$

(1 point)

Integrating both sides gives

$$\log|\log(u)| = \log|x| + C$$

(3 points)

Note: the differential equation is only defined for $x > 0$ so we can omit the absolute value bars in the right hand side. First solving for u and then for y gives

$$u = e^{Kx} \quad \Rightarrow \quad y = xe^{Kx}$$

where $K = \pm e^{C}$ is a new arbitrary constant.
(3 points)

Finally, the initial condition $y(1) = e$ gives $K = 1$ so that $y = xe^{x}$.
(2 points)

Solution of Problem 2 (10 points)

This is a Bernoulli equation with $\alpha = 4$. Therefore, we introduce the new variable $z = y^{1-\alpha} = y^{-3}$.
(1 point)

The new variable satisfies a linear differential equation:

$$\frac{dz}{dx} - 6xz = 3x$$

(3 points)

Multiplying with the integrating factor e^{-3x^2} gives

$$e^{-3x^2}\frac{dz}{dx} - 6xe^{-3x^2}z = 3xe^{-3x^2} \quad \Rightarrow \quad \frac{d}{dx}\left[e^{-3x^2}z\right] = 3xe^{-3x^2}$$

Integrating both sides gives

$$e^{-3x^2}z = -\tfrac{1}{2}e^{-3x^2} + C \quad \Rightarrow \quad z = -\tfrac{1}{2} + Ce^{3x^2}$$

(5 points)

Finally, we obtain the solution for the original problem:

$$z = \frac{1}{y^3} \quad \Rightarrow \quad y = \frac{1}{\sqrt[3]{z}} = \frac{1}{\sqrt[3]{-\frac{1}{2} + Ce^{3x^2}}}$$

(1 point)

Solution of Problem 3 (2 + 12 + 6 points)

(a) Expanding the determinant along the first row gives

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 1 & -\lambda & 2 \\ 1 & -1 & 2-\lambda \end{bmatrix} \\
&= (1-\lambda)\det \begin{bmatrix} -\lambda & 2 \\ -1 & 2-\lambda \end{bmatrix} + \det \begin{bmatrix} 1 & -\lambda \\ 1 & -1 \end{bmatrix} \\
&= (1-\lambda)(\lambda^2 - 2\lambda + 2) + (\lambda - 1) \\
&= (1-\lambda)(\lambda^2 - 2\lambda + 1) \\
&= (1-\lambda)(\lambda - 1)^2 \\
&= (1-\lambda)^3
\end{aligned}$$

(2 points)

(b) The generalized eigenspaces of A are given by:

$$A - I = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad E_\lambda^1 = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(2 points)

$$(A - I)^2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad E_\lambda^2 = \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(2 points)

$$(A - I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad E_\lambda^3 = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(2 points)

Therefore, the dot diagram is given by

$$\left.\begin{aligned}
r_1 &= \dim E_\lambda^1 = 1 \\
r_2 &= \dim E_\lambda^2 - \dim E_\lambda^1 = 1 \\
r_3 &= \dim E_\lambda^3 - \dim E_\lambda^2 = 1
\end{aligned}\right\} \quad \Rightarrow \quad \begin{matrix} \bullet \\ \bullet \\ \bullet \end{matrix}$$

which means we have one cycle of length three. In particular, we obtain

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(2 points)

To construct the matrix Q we start by selecting a vector $\mathbf{v} \in E_\lambda^3$ which is not contained in the previous eigenspaces. For example, we can take

$$\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad (A - I)\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad (A - I)^2\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(2 points)

Listing these vectors in the reverse order gives the following matrix

$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(2 points)

(c) A possible fundamental matrix is given by

$$Y(t) = Qe^{Jt} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$$

(4 points)

Writing out the product gives

$$Y(t) = e^t \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} = e^t \begin{bmatrix} 1 & 1+t & t+\frac{1}{2}t^2 \\ 1 & 2+t & 2t+\frac{1}{2}t^2 \\ 0 & 1 & 1+t \end{bmatrix}$$

(2 points)

Note that it was not necessary to compute Q^{-1} in this problem. The question was to compute a fundamental matrix, and not to compute e^{At}!

Solution of Problem 4 (5 + 5 + 5 + 5 points)

(a) Apply the Mean Value Theorem to the function $f(y) = \sqrt{1+y^2}$: for all $y, z \in \mathbb{R}$ there exists a point c between y and z such that

$$f(y) - f(z) = f'(c)(y-z) \quad \Rightarrow \quad \sqrt{1+y^2} - \sqrt{1+z^2} = \frac{c}{\sqrt{1+c^2}}(y-z)$$

(3 points)

Taking absolute values gives

$$\left|\sqrt{1+y^2} - \sqrt{1+z^2}\right| = \left|\frac{c}{\sqrt{1+c^2}}\right| \cdot |y-z| \leq |y-z|$$

(2 points)

The inequality follows from the fact that

$$\sqrt{1+c^2} \geq \sqrt{c^2} = |c| \quad \Rightarrow \quad \left|\frac{c}{\sqrt{1+c^2}}\right| = \frac{|c|}{\sqrt{1+c^2}} \leq 1$$

(b) Let $y, z \in C([0,b])$ and $x \in [0,b]$ be arbitrary. Then the triangle inequality for integrals and part (a) together imply that

$$\begin{aligned}
\left|(Ty)(x) - (Tz)(x)\right| &= \left|\int_0^x \sqrt{1+y(t)^2} - \sqrt{1+z(t)^2}\,dt\right| \\
&\leq \int_0^x \left|\sqrt{1+y(t)^2} - \sqrt{1+z(t)^2}\right| dt \\
&\leq \int_0^x \left|y(t) - z(t)\right| dt
\end{aligned}$$

(3 points)

Noting that $|y(t)-z(t)|e^{-\alpha t} \leq \|y-z\|$ for all $t \in [0,b]$ gives

$$\begin{aligned}
\left|(Ty)(x) - (Tz)(x)\right| &\leq \int_0^x \left|y(t) - z(t)\right| e^{-\alpha t} e^{\alpha t}\,dt \\
&\leq \|y-z\| \int_0^x e^{\alpha t}\,dt \\
&= \frac{e^{\alpha x} - 1}{\alpha}\|y-z\|
\end{aligned}$$

(2 points)

(c) From part (b) it follows that for all $x \in [0,b]$ we have that

$$\left|(Ty)(x) - (Tz)(x)\right| e^{-\alpha x} \leq \frac{1 - e^{-\alpha x}}{\alpha}\|y-z\| \leq \frac{1}{\alpha}\|y-z\|$$

Now taking the supremum over all $x \in [0,b]$ gives the desired result.
(5 points)

(d) Recall Banach's fixed point theorem: assume that

(i) B is a Banach space

(ii) $D \subset B$ is closed

(iii) $T : D \subset B \to B$ satisfies $T(D) \subset D$ and

$$\exists\, 0 < q < 1 \quad \text{such that} \quad \|T(y) - T(z)\| \leq q\|y - z\| \quad \forall\, y,\, z \in D$$

Then there exists a unique point $\bar{y} \in B$ such that $T(\bar{y}) = \bar{y}$. Moreover, iterations of T converge to this fixed point:

$$y_0 \in D, \quad y_{n+1} = T(y_n) \quad \Rightarrow \quad \lim_{n\to\infty} y_n = \bar{y}$$

We apply this theorem with $B = D = C([0, b])$. Then D is automatically closed and $T(D) \subset D$ is trivially satisfied. For $\alpha > 1$ we have $q = 1/\alpha \in (0, 1)$ so that T becomes a contraction. Therefore, Banach's fixed point theorem implies that there exists a unique $y \in C([0, b])$ such that $T(y) = y$.

Since we have the following equivalences

$$T(y) = y \quad \Leftrightarrow \quad y(x) = \int_0^x \sqrt{1 + y(t)^2}\, dt \quad \Leftrightarrow \quad \begin{cases} y' &= \sqrt{1 + y^2} \\ y(0) &= 0 \end{cases}$$

we obtain the existence and uniqueness result for the initial value problem.
(5 points)

Solution of Problem 5 (3 + 10 points)

(a) We have

$$u = e^{x^2}, \qquad u' = 2xe^{x^2}, \qquad u'' = (4x^2+2)e^{x^2}$$

Substitution into the differential equation gives

$$(4x^2+2)e^{x^2} - 8x^2e^{x^2} + (4x^2-2)e^{x^2} = (4x^2 - 8x^2 + 4x^2 + 2 - 2)e^{x^2} = 0$$

(3 points)

(b) We have

$$u_2 = c(x)e^{x^2}, \qquad u_2' = [c'(x)+2xc(x)]e^{x^2}, \qquad u_2'' = [c''(x)+4xc'(x)+(4x^2+2)c(x)]e^{x^2}$$

(4 points)

Substitution in the differential equation gives

$$e^{x^2}c''(x) = 0 \quad \Rightarrow \quad c''(x) = 0 \quad \Rightarrow \quad c(x) = ax + b$$

Hence, a second solution is given by $u_2 = (ax+b)e^{x^2}$.
(4 points)

The Wronskian determinant of u_1 and u_2 is given by

$$W = u_1u_2' - u_1'u_2 = ae^{2x^2}$$

To make u_1 and u_2 linearly independent we need to take $a \neq 0$. An obvious choice is $a = 1$ and $b = 0$.
(2 points)

Since $u_2(x) = c(x)u_1$ an alternative argument is that u_1 and u_2 are linearly independent if and only if $c(x)$ is not a constant function, which is the case if $a \neq 0$.

Solution of Problem 6 (10 + 5 points)

(a) First we solve the homogeneous differential equation:

$$u'' - u = 0 \quad \Rightarrow \quad u = c_1 e^x + c_2 e^{-x}$$

(2 points)

The solution $u_1 = e^x - e^{-x}$ satisfies the left boundary condition $u(0) = 0$.
(1 point)

The solution $u_2 = e^x - e^2 e^{-x}$ satisfies the right boundary condition $u(1) = 0$.
(1 point)

Their Wronskian determinant is given by

$$W = u_1 u_2' - u_1' u_2 = 2(e^2 - 1)$$

(2 points)

Since $p(x) = 1$ in this problem the Green's function is given by

$$\Gamma(x, \xi) = \frac{1}{2(e^2 - 1)} \begin{cases} (e^x - e^{-x})(e^\xi - e^{2-\xi}) & \text{if } 0 \le x \le \xi \le 1 \\ (e^\xi - e^{-\xi})(e^x - e^{2-x}) & \text{if } 0 \le \xi \le x \le 1 \end{cases}$$

(4 points)

(b) The solution of the boundary value problem with $f(x) = e^{-x}$ is given by

$$u(x) = \int_0^1 \Gamma(x, \xi) f(\xi)\, d\xi$$

(1 point)

which can be written as

$$u(x) = \frac{e^x - e^{-x}}{2(e^2 - 1)} \int_x^1 1 - e^{2-2\xi}\, d\xi + \frac{e^x - e^{2-x}}{2(e^2 - 1)} \int_0^x 1 - e^{-2\xi}\, d\xi$$

Computing the integrals gives

$$\int_x^1 1 - e^{2-2\xi}\, d\xi = \frac{3}{2} - \frac{e^{2-2x}}{2} - x$$

(2 points)

$$\int_0^x 1 - e^{-2\xi}\, d\xi = -\frac{1}{2} + \frac{e^{-2x}}{2} + x$$

(2 points)

Putting everything together and simplifying the result gives

$$u(x) = \frac{e^x - e^{-x}}{2(e^2 - 1)} - \frac{x e^{-x}}{2}$$